A STRUCTURE THEOREM FOR THE GROMOV-WITTEN INVARIANTS OF KÄHLER SURFACES

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Abstract

We prove a structure theorem for the Gromov-Witten invariants of compact Kähler surfaces with geometric genus $p_g > 0$. Under the technical assumption that there is a canonical divisor that is a disjoint union of smooth components, the theorem shows that the GW invariants are universal functions determined by the genus of this canonical divisor components and the holomorphic Euler characteristic of the surface. We compute special cases of these universal functions.

Much of the work on the Gromov-Witten invariants of Kähler surfaces has focused on rational and ruled surfaces, which have geometric genus $p_g = 0$. This paper focuses on surfaces with $p_g > 0$, a class that includes most elliptic surfaces and most surfaces of general type. In this context we prove a general "structure theorem" that shows (with one technical assumption) how the GW invariants are completely determined by the local geometry of a generic canonical divisor.

The structure theorem is a consequence of a simple fact: the "Image Localization Lemma" of Section 3. Given a Kähler surface X and a canonical divisor $D \in |K_X|$, this lemma shows that the complex structure J on X can be perturbed to a *non-integrable* almost complex structure J_D with the property that the image of all J_D-holomorphic maps lies in the support of D. This immediately gives some striking vanishing theorems for the GW invariants of Kähler surfaces (see Section 3). More importantly, it implies that the Gromov-Witten invariant of X for genus g and n marked points is a sum

$$GW_{g,n}(X, A) = \sum GW_{g,n}^{\mathrm{loc}}(D_k, A_k)$$

over the connected components D_k of D of "local invariants" that count the contribution of maps whose image lies in or (after perturbing to a generic moduli space) near D_k. These local invariants have not been previously defined. The proof of their existence relies on using non-integrable structures and geometric analysis techniques.

Our structure theorem characterizes the local invariants by expressing them in terms of usual GW invariants of certain standard surfaces. For this we make the mild assumption that one can deform the Kähler structure on X and choose a canonical divisor D so that all components of D are smooth. With this assumption, the restriction of K_X to each component D_k of multiplicity m_k is the normal bundle N_k to D_k, and

$$N_k^{m_k+1} = K_{D_k},$$

that is, N_k is a holomorphic $(m_k + 1)$-th root of the canonical bundle of the curve D_k. The local invariants are given by universal functions

(0.1) $$L^i(t) \in \prod_{g,n} H_*(\overline{\mathcal{M}}_{g,n} \times D_k^n)\,[[t, \lambda]],$$

depending only on topological invariants of the pair (D_k, N_k). Four types of such universal functions are relevant. Each is defined in terms of the GW invariant of a rational surface or a local GW invariant of a line bundle over a curve (they can also be defined using obstruction bundles, but we do not pursue that approach here). There is one universal function $L^0(t)$ for exceptional curves. This enters into the blow-up formula for GW invariants proved in Section 5. The blow-up formula reduces the computation of the GW invariants to the case of minimal surfaces. In light of the vanishing results given in Section 3, there are only two types of minimal surfaces to consider: properly elliptic surfaces and surfaces of general type.

A minimal properly elliptic surface can be deformed to guarantee the existence of a canonical divisor whose support is the union of smooth elliptic fibers. The structure theorem separates these into two types: regular fibers and multiple fibers with multiplicity $m \geq 2$; the corresponding universal functions are $L^1(t)$ and $L_m^2(t)$ respectively. For a minimal surface of general type all canonical divisors are connected; we assume that one such divisor D is smooth and reduced. By the adjunction formula, D has genus $h = K_X^2 + 1$. The GW invariant is then given by one of two universal functions $L_{h,\pm}^3(t)$ for this h.

The GW invariants of a Kähler surface X can be regarded as a power series in formal variables t_A with $A \in H_2(X; \mathbb{Z})$ as described in Section 1. Each smooth component D_k of the canonical class, we can replace t by t_{D_k} in the appropriate universal function (0.1), taking h to be the genus D_k. Pushing forward under the map $(\iota_{D_k})_* : H_*(\overline{\mathcal{M}}_{g,n} \times D_k^n) \to H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$ induced by the inclusion $D_k \hookrightarrow X$ then gives a series (suppressing $(\iota_{D_k})_*$ in the notation)

$$L^i(t_{D_k}) \in \prod_{g,n} H_*(\overline{\mathcal{M}}_{g,n} \times X^n)[[t_{D_k}, \lambda]],$$

which gives the local contribution of D_k to the GW series. For each surface, only a few of these series are needed. The structure theorem

lists the possibilities. The contribution GW_X^0 of the class $A = 0$ (that is, the evaluation of (1.4) at $t = 0$) must be separated out. Note that GW_X^0 has been explicitly computed (see [**16**]). Also note that surfaces with $p_g > 0$ have a unique minimal model ([**4**], p. 243).

Theorem 0.1 (Structure Theorem). *Let X be a closed Kähler surface with $p_g > 0$ and smooth canonical divisor D. Write $D = \sum_i E_i + D'$ where $\{E_i\}$ are the exceptional curves in D. Then the GW invariant of X is a sum*

$$GW_X = GW_X^0 + \sum_{E_i} L^0(t_{E_i}) + GW_X',$$

where GW_X' is given as follows according to the type of the minimal model X' of X:

1) *If X' is K3 or abelian, then $GW_X' = 0$.*
2) *If $\pi : X' \to C$ is properly elliptic, we can assume that the canonical divisor D' has the form $\sum n_j F_j + \sum (m_k - 1)F_k$ for regular fibers F_j and smooth multiple fibers F_k of multiplicity m_k. We then have*

$$GW_X' = k_\pi \, L^1(t_F) + \sum_k L_{m_k}^2(t_{F_k}),$$

   where F is a regular fiber, $t_{F_k}^{m_k} = t_F$, and $k_\pi = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C)$.
3) *If X' is general type and we can choose D' to be smooth with multiplicity 1, then D' has genus $h = K_{X'}^2 + 1 \geq 2$ and*

$$GW_X' = \begin{cases} L_{h,+}^3(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is even} \\ L_{h,-}^3(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is odd.} \end{cases}$$

A more detailed version of the structure theorem is given, with proofs, in Sections 4–7. Section 8 contains some analytic results about the linearization of the J_α-holomorphic map equation, which has some remarkable properties. Those are used in Sections 9 and 10 to explicitly compute the contribution to the GW invariants of special types of covers. We do this for double covers, then for all etale covers of elliptic fibers.

The structure theorem shows that, under the stated hypotheses, the GW invariants are determined by the map $H^*(X, \mathbb{Q}) \to H^*(D, \mathbb{Q})$ induced by the inclusion $D \subset X$ and by the parity of the holomorphic Euler characteristic, which is given in terms of the Betti numbers by $\chi(\mathcal{O}_X) = \frac{1}{2}(1 - b^1 + b^+)$. In the case when X is a simply-connected surface of general type with a smooth reduced canonical divisor, this information is determined by the homology and Seiberg-Witten invariants of X, and hence depends only on the differentiable structure of X. Furthermore, the SW invariants are equivalent to Taubes' Gr invariants, which correspond to a subset of GW invariants [**23**], [**12**]. But we learn

from the structure theorem that the full set of GW invariants contain exactly the same information as the Gr and SW invariants.

Our structure theorem applies for surfaces with $p_g > 0$. This is exactly the case when the dimension of the spaces of stable maps differs from the dimension of the generalized Severi variety, and thus the GW invariants are not enumerative invariants. The J_α-holomorphic map equation can also be used to define a set of "Family GW invariants" that are directly related to enumerative invariants. That context is explained in [17], [18] and [19].

One might hope that the structure theorem extended to non-Kähler symplectic manifolds with $b^+ > 1$. Unfortunately, as M. Usher observed ([24], page 4), McMullen and Taubes have constructed a symplectic four-manifold whose GW invariant is not the sum of local invariants supported on the components of the canonical class.

1. Gromov-Witten invariants

We will use the definitions and notation of [14] for stable maps and the Gromov-Witten invariants; these are based on the approach developed by Ruan-Tian [22] and Li-Tian [20]. In summary, the key definitions go as follows. A *bubble domain* B is a finite connected union of smooth oriented 2-manifolds B_i joined at nodes together with n marked points, none of which are nodes. Collapsing the unstable components to points gives a connected domain $st(B)$ with some arithmetic genus g. Let $\overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve over the Deligne-Mumford space of genus g curves with n marked points. We can put a complex structure j on B by specifying an orientation-preserving map $\varphi_0 : st(B) \to \overline{\mathcal{U}}_{g,n}$, which is a diffeomorphism onto a fiber of $\overline{\mathcal{U}}_{g,n}$. We will often write C for the curve (B, j). A (J, ν)-*holomorphic map* from B is then a map $(f, \varphi) : B \to X \times \overline{\mathcal{U}}_{g,n}$ where $\varphi = \varphi_0 \circ st$ and which satisfies

$$\bar{\partial}_J f = \varphi^* \nu$$

(here the perturbation ν is a tensor on $X \times \overline{\mathcal{U}}_{g,n}$; see [14]). Such a map is a *stable map* if the restriction of (f, φ) to each component of B is non-trivial in homology. For generic (J, ν) the moduli space $\mathcal{M}_{g,n}(X, A)$ of stable (J, ν)-holomorphic maps representing a class $A \in H_2(X)$ is a smooth orbifold of (real) dimension

$$(1.1) \qquad -2K_X \cdot A + (\dim X - 6)(1 - g) + 2n.$$

Its compactification carries a (virtual) fundamental class whose push-forward under the map

$$(1.2) \qquad \overline{\mathcal{M}}_{g,n}(X,A) \xrightarrow{\text{st}\times\text{ev}} \overline{\mathcal{M}}_{g,n} \times X^n$$

defined by stabilization and evaluation at the marked points is the Gromov-Witten invariant

$$GW_{g,n}(X,A) \in H_*(\overline{\mathcal{M}}_{g,n} \times X^n).$$

This is equivalent to the collection of "GW numbers"

$$(1.3) \qquad GW_{g,n}(X,A)(\mu; \gamma^1, \ldots, \gamma^n)$$

obtained by evaluating on classes $\mu \in H^*(\overline{\mathcal{M}}_{g,n})$ and $\gamma^j \in H^*(X)$ whose total degree is the dimension (1.1) of the space of stable maps. The number (1.3) is obtained by choosing (generic) geometric representatives $M \subset \overline{\mathcal{M}}_{g,n}$ and Γ_i of the classes Poincaré dual to $\mu \in H^*(\overline{\mathcal{M}}_{g,n})$ and $\gamma^i \in H^*(X)$ and counting, with sign, the finite set of maps $f : C \to X$ in $st(C) \in M$ and $f(x_i) \in \Gamma_i$ for each marked point x_i.

It is convenient to assemble these into a single invariant by introducing variables λ to keep track of the Euler class and t_A satisfying $t_A t_B = t_{A+B}$ to keep track of A. The GW series of (X, ω) is then the formal series

$$(1.4) \qquad GW_X = \sum_{A,g,n} \frac{1}{n!} GW_{g,n}(X,A) \; t_A \; \lambda^{2g-2}.$$

2. J_α-holomorphic maps into Kähler surfaces

Fix a Kähler surface (X, J, g). On X, holomorphic sections of the canonical bundle are holomorphic $(2,0)$ forms, and the dimension of the space $H^{2,0}(X)$ of such forms is the geometric genus p_g of X. We will always assume that $p_g > 0$. Each $\alpha \in H^{2,0}(X)$ can be identified with an element of the $2p_g$-dimensional real vector space

$$\mathcal{H} = \text{Re}\big(H^{2,0} \oplus H^{0,2} \big).$$

Using the metric, each $\alpha \in \mathcal{H}$ defines an endormorphism K_α of TX by the equation

$$(2.1) \qquad \langle u, K_\alpha v \rangle = \alpha(u, v).$$

These endomorphisms K_α are central to our discussion, and we will frequently use the following properties. Denote by ∇ the Levi-Civita connection of the given metric.

Lemma 2.1. *The K_α are skew-adjoint and anti-commute with J ($K_\alpha J = -J K_\alpha$). Furthermore,*

$$\text{(a)} \;\; \nabla K_\alpha = K_{\nabla \alpha} \qquad \text{and} \qquad \text{(b)} \;\; K_\alpha^2 = -|\alpha|^2 Id.$$

Consequently, they satisfy the pointwise Clifford relations

$$K_\alpha K_\beta + K_\beta K_\alpha = -2\langle \alpha, \beta \rangle \, Id.$$

Proof. The first two statements and (a) are immediate from (2.1). The Clifford relations follow by polarization from (b), which is easily proved (cf. [**17**]).   q.e.d.

Now consider holomorphic maps $f : C \to X$ from a connected complex curve with complex structure j into X. It is standard in geometric analysis to consider solutions of the perturbed J-holomorphic map equation

$$\overline{\partial}_J f = \nu,$$

where $\overline{\partial}_J f = \frac{1}{2}(df + Jdfj)$ and where ν is an appropriate perturbation term. In [**17**] the first author observed that, on a Kähler surface with $p_g > 0$, there is a natural family of such perturbations parameterized by $\mathcal{H}$. Specifically, we can consider the pairs (f, α) satisfying

(2.2) $$\overline{\partial}_J f = K_\alpha \partial_J f j.$$

This can equally well be viewed as a set of unperturbed holomorphic map equations for a family of almost complex structures $\{J_\alpha\}$ parameterized by $\mathcal{H}$. For each $\alpha \in \mathcal{H}$ the endomorphism JK_α is skew-adjoint, so $Id + JK_\alpha$ is injective, and hence invertible. Thus there is a family of almost complex structures

(2.3) $$J_\alpha = (Id + JK_\alpha)^{-1} J \, (Id + JK_\alpha)$$

on X parameterized by $\alpha \in \mathcal{H}$. A simple computation shows that (2.2) is equivalent to the J_α-holomorphic map equation

(2.4) $$\overline{\partial}_{J_\alpha} f = 0$$

for maps $f : C \to X$. Our structure theorem for GW invariants will emerge from studying the solutions of this equation for a fixed $\alpha \in \mathcal{H}$. Note that while α itself is holomorphic, the corresponding almost complex structure J_α need not be integrable. On the other hand, J_α is generally not a generic almost complex structure on X, so the moduli space of J_α holomorphic maps does not directly define the GW invariants.

3. The Localization Lemma and vanishing results

The discussion in this section builds on the following simple principle about Gromov-Witten invariants.

Vanishing Principle 3.1. If for *some* ω-tamed almost complex structure J, a class $A \in H_2(X)$ cannot be represented by a J-holomorphic curve of genus g, then $GW_{g,n}(X, A)$ vanishes.

The proof is straightforward: if some $GW_{g,n}(X, A)$ were not zero, we could choose sequences $\{J_n\}$ of generic almost complex structures converging to J for which there were J_n-holomorphic maps representing A. But then, by the compactness theorem for pseudo-holomorphic maps, a subsequence of those maps would limit to a J-holomorphic map representing A, contradicting the assumption. As a simple application, note that for a Kähler surface (X, J), any J-holomorphic curve represents a $(1,1)$ class, so $GW_{g,n}(X, A) = 0$ unless A is a $(1,1)$ class. This observation allows us to restrict attention to $(1,1)$ classes for all our results.

Lemma 3.2 (Image Localization Lemma). *Fix a Kähler surface (X, J) with $p_g > 0$ and $\alpha \in \mathcal{H}$. If $f : C \to X$ is a J_α-holomorphic map with connected domain that represents a $(1, 1)$ class $A \neq 0$, then f is in fact a J-holomorphic map whose image $f(C)$ lies in the support of the zero divisor D_α of α.*

Proof. For any C^1 map $f : C \to X$ we have the pointwise equality

$$(3.1) \qquad \langle \overline{\partial} f, \ K_\alpha \partial f j \rangle \ d\mathrm{vol} = f^* \alpha$$

(see Proposition 1.3 of [**17**]). Integrating over the domain and using (2.2) gives

$$\int_C |\overline{\partial} f|^2 = \int_C \langle \overline{\partial} f, \ K_\alpha \partial f j \rangle = \int_C f^* \alpha.$$

Because α is closed, the last integral is the homology pairing $\alpha[A]$. This vanishes on the $(1,1)$ class A because α is a linear combination of $(2, 0)$ and $(0, 2)$ forms. Thus $\overline{\partial} f \equiv 0$ on C. Then using (2.2), Lemma 2.1 and the equality $|df|^2 = |\overline{\partial} f|^2 + |\partial f|^2$, we obtain

$$(3.2) \qquad 0 \ = \ \int_C |\overline{\partial} f|^2 \ = \ \int_C |K_\alpha \partial f j|^2 \ = \ \int_C |\alpha|^2 \, |df|^2.$$

Since $A \neq 0$, there is at least one irreducible component of C with $df \not\equiv 0$. On each such component C_i, df has finitely many zeros, so (3.2) implies that $f(C_i)$ lies in the support of D_α. Each of the remaining components is taken to a single point by f; since C is connected those points also lie in the support of D_α. q.e.d.

Lemma 3.2 leads directly to some striking vanishing results for GW invariants. For example, K3 and abelian surfaces have trivial canonical bundle, so they admit $(2,0)$ forms that vanish nowhere. Lemma 3.2 and Principle 3.1 then give:

Corollary 3.3. *For K3 and abelian surfaces, all GW invariants $GW_{g,n}(X, A)$ vanish for $A \neq 0$.*

We also obtain a vanishing result for the GW numbers (1.3). This follows from the Vanishing Principle and the geometric interpretation of the GW numbers.

Corollary 3.4. *On a Kähler surface X with $p_g > 0$, any GW invariant constrained to pass through (generic) points or circles vanishes. Equivalently, $GW_{g,n}(X,A)(\mu;\gamma^1,\ldots,\gamma^k) = 0$ whenever one of the γ^j lies in $H^3(X)$ or $H^4(X)$.*

Proof. When $PD(\gamma^j)$ is a point or 1-dimensional class, we can fix a geometric representative Γ_j disjoint from D_α. Then, if the invariant $GW_{g,n}(X,A)(\gamma^1,\ldots,\gamma^k)$ were not zero, we could find a sequence $\{J_n\}$ of generic almost complex structures converging to J_α and J_n-holomorphic maps $\{f_n\}$ representing A with $f_n(x_i) \in \Gamma_i$ for all i and n. The compactness theorem would then yield a limit J_α-holomorphic map f satisfying $f(x_j) \in \Gamma_j$, contradicting Lemma 3.2.   q.e.d.

The Image Localization Lemma allows us to localize the GW invariants for Kähler surfaces with $p_g > 0$. When X is such a surface and $\alpha \in \mathcal{H}$, the support of the zero divisor D_α of α is a union of disjoint topological components D_α^k. Lemma 3.2 implies that, for generic (J,ν) near $(J_\alpha, 0)$, the image of any (J,ν)-holomorphic map with connected domain lies in an open neighborhood U_k of one and only one of the D_α^k. Thus the compactified moduli space of (J,ν)-holomorphic maps representing a non-zero class A is a disjoint union

$$(3.3) \qquad \overline{\mathcal{M}}_{g,n}(X,A) = \coprod \overline{\mathcal{M}}_{g,n}(U_k, A_k)$$

over all A_k with $(\iota_k)_* A_k = A$ under the inclusion $\iota_k : U_k \to X$. Note that each U_k is an open symplectic four-manifold with $H_*(U_k) = H_*(D_\alpha^k)$. As in Section 1, the image of each $\overline{\mathcal{M}}_{g,n}(U_k, A_k)$ under the map (1.2) defines a homology class

$$(3.4) \qquad GW_{g,n}^{\mathrm{loc}}(D_\alpha^k, A_k) \in H_*(\overline{\mathcal{M}}_{g,n} \times D_k^n)$$

that we call the *local GW invariant of D_α^k for the (non-zero) class A_k*. These local invariants depend on the choice of the canonical divisor D_α, rather than on the choice of α itself. Indeed, if $\beta \in \mathcal{H}$ also has zero divisor D_α, then $\beta = c\,\alpha$ for some constant. Thus, J_α and J_β are connected by a path $J_t = J_{\alpha_t}$ with $\alpha_0 = \alpha$ and $\alpha_1 = \beta$, for which every J_t-holomorphic map lies in the support of D_α. The standard corbodism argument then shows that the local invariants $GW_{g,n}^{\mathrm{loc}}(D_\alpha^k, A_k)$ associated with J_α and J_β are the same.

We remark in passing that the local invariants (3.4) can also be regarded as elements of the homology of the space $\overline{\mathcal{M}}_{g,n}(D_\alpha^k, d_k(A))$ of stable maps into the curve D_α^k with degree determined by the equation $(\iota_k)_* A_k = d_k(A)[D_\alpha^k]$. From that perspective, (3.4) is the image of the local invariant under the homology map induced by the evaluation map (1.2) with $X = D_\alpha^k$.

Pushing (3.3) forward under the evaluation map (1.2) and passing to homology shows that, for $A \neq 0$,

$$(3.5) \qquad GW_{g,n}(X, A) = \sum_{(\iota_k)_* A_k = A} GW_{g,n}^{\mathrm{loc}}(D_\alpha^k, A_k)$$

for any choice of the canonical divisor D_α. This formula is the first step toward our structure theorem. It shows that the GW invariants can be expressed as a sum of local contributions associated with the components of a canonical divisor.

4. Local GW invariants

The local invariants in the sum (3.5) depend, at least *a priori*, on the local geometry of J_α around the components of the canonical divisor D_α. In the rest of this paper we will write $D_\alpha = \sum m_k D_k$ and assume that the D_k are smooth and disjoint. We will show that the local invariants depend only on discrete data g, n, d and the multiplicities m_k. When D_k is smooth every map with image in D_k represents a multiple d of $[D_k]$, so we will write the local invariant (3.4) as

$$GW_{g,n}^{\mathrm{loc}}(D_k, m_k, d)$$

or simply $GW_{g,n}^{\mathrm{loc}}(D_k, d)$ when $m_k = 1$. Then, for $A \neq 0$, equation (3.5) reads

$$(4.1) \qquad GW_{g,n}(X, A) = \sum_{d_k[D_k] = A} GW_{g,n}^{\mathrm{loc}}(D_k, m_k, d_k).$$

Using arguments like those in the previous section, one can also define local GW invariants of some open complex surfaces. Fix a smooth curve D with canonical bundle K_D and a line bundle $\pi : N \to D$ satisfying $N^{m+1} = K_D$. The total space of N is a complex manifold; from the exact sequence $0 \to \pi^* N \to TN \to \pi^* TD \to 0$ we see that its canonical bundle is

$$(4.2) \quad K_N = \wedge^2 T^* N = \pi^* K_D \otimes \pi^* N^* = \pi^* N^{m+1} \otimes \pi^* N^* = \pi^* N^m.$$

The bundle $\pi^* N$ has a tautological section σ whose zero divisor is exactly D. Then $\alpha = \sigma^m$ is a section of the canonical bundle K_N, and so is a holomorphic (2,0)-form on N. The argument used to prove Lemma 3.2 then shows that the image of any J_α-holomorphic map into N lies in D. On the other hand, an open neighborhood U of $D \subset N$ is isomorphic to some open neighborhood V of the zero section D_0 in the projectivization $\mathbb{P} = \mathbb{P}(N \oplus \mathcal{O}_D)$ by an isomorphism taking D to D_0. The pull-back of the Kähler form on V by that isomorphism gives a Kähler form on U. Thus, for any generic (J, ν) sufficiently close to $(J_\alpha, 0)$, the moduli space $\mathcal{M}_{g,n}(U, d[D])$ can be compactified by standard geometric analysis

techniques. Taking the image as in (1.2) yields homology classes

(4.3) $$L_{g,n}(N, m, d) \in H_*(\overline{\mathcal{M}}_{g,n} \times D^n)$$

that we call the *local GW invariants of N associated with mD* for maps representing $d[D]$, $d > 0$. When $m = 1$ we will often write (4.3) as simply $L_{g,n}(N, d)$. These local invariants depend on the zero divisor of α but not on α itself by the following reasoning. Let β be a section of the canonical bundle K_N, defined on a neighborhood U of $D \subset N$, such that the zero divisor of β is mD. Then $\beta = h\,\alpha$ for some holomorphic function h whose restriction of h to D is a non-zero constant. Hence, after shrinking U if necessary, J_a and J_β can be connected by a path J_{α_t} where the zero divisor of each α_t on U is mD. As in the previous section, the usual corbordism argument then shows that the local invariants associated with J_α and J_β are the same.

A similar corbodism argument gives the following fact.

Lemma 4.1. *If $\{(N_t, D_t)\}_{0 \le t \le 1}$ is a smooth path of line bundles satisfying $N_t^{m+1} = K_{D_t}$ then*

$$L_{g,n}(N_0, m, d) = L_{g,n}(N_1, m, d).$$

Thus the local invariants (4.3) depend only on the discrete data g, n, and d and the deformation class of the pair (N, D).

Example 4.2. Consider the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$. The complex structure J_0 on the total space of $\mathcal{O}(-1)$ is not of the form J_α, but nevertheless has the property that any J_0-holomorphic map representing the class $d[\mathbb{P}^1]$ has an image in the zero section in the total space of the bundle $\mathcal{O}(-1)$. The argument used above thus applies for J_0 as well as for J_α, showing that J_0 itself defines the local GW invariants $L_{g,n}(\mathcal{O}(-1), d)$.

We can relate the local invariants of D_k defined in (3.4) with the local invariants of its normal bundle defined in (4.3), as follows.

Lemma 4.3. *Let X be a Kähler surface with $p_g > 0$ and $D_\alpha = \sum m_k D_k$ be the zero divisor of $\alpha \in \mathcal{H}$. If D_k is smooth with normal bundle N_k and $D_k \cap D_\ell = \emptyset$ for all $\ell \ne k$, then*

$$GW_{g,n}^{\mathrm{loc}}(D_k, m_k, d) = L_{g,n}(N_k, m_k, d).$$

Proof. Fix $D = D_k$. By the adjunction formula, the normal bundle N of D satisfies $N^{m+1} = K_D$ with $m = m_k$. Let Z be the blow-up of $X \times \mathbb{C}$ along $D \times \{0\}$. The projection $X \times \mathbb{C} \to \mathbb{C}$ lifts to a map $p : Z \to \mathbb{C}$ whose fibers $Z_\lambda = p^{-1}(\lambda)$ are isomorphic to X for $\lambda \ne 0$ and whose central fiber Z_0 is a singular surface $X \cup_D \mathbb{P}$ where $\mathbb{P}$ is the ruled surface $\mathbb{P}(N \oplus \mathcal{O}_D) \to D$ defined by fiber projectivization. The proper transform of $D \times \mathbb{C}$ is a smooth divisor $\tilde{D} \subset Z$, disjoint from the proper transforms $\tilde{D}_\ell$ of the other $D_\ell \times \mathbb{C}$, and α gives rise to a section $\tilde{\alpha}$ of the

canonical bundle K_Z of Z whose zero divisor is $m\tilde{D} + \sum_{\ell \neq k} m_\ell \tilde{D}_\ell + \mathbb{P}$. Now fix a tubular neighborhood U of $\tilde{D}$ that is disjoint from the $\tilde{D}_\ell$. Let κ be the line bundle of the divisor $m\tilde{D}$, and let $\beta \in \Gamma(\kappa)$ be a section with zero divisor $m\tilde{D}$. For each λ, the intersection $U_\lambda = U \cap Z_\lambda$ is a tubular neighborhood of $D_\lambda = \tilde{D} \cap Z_\lambda$. The restriction

$$\kappa_\lambda \;=\; \kappa|_{U_\lambda}$$

is the line bundle on U_λ with divisor mD_λ. Observe that:

- For $\lambda \neq 0$, the normal bundle N_λ to Z_λ in Z is trivial. Restricting the exact sequence $0 \to TZ_\lambda \to TZ \to N_\lambda \to 0$ to U_λ then shows that the canonical bundle of U_λ is the restriction of the canonical bundle of Z, which is the bundle of the divisor $m\tilde{D} \cap U_\lambda = mD_\lambda$.
- For $\lambda = 0$ we use a different argument. By the definition of blow-up, U_0 is biholomorphic to a neighborhood of the zero section in the total space of the bundle $N \to D$; in fact, this identifies the zero section with D_0. But by (4.2) the canonical bundle of N has a tautological section whose divisor is m times that zero section.

Thus, κ_λ is the canonical bundle of U_λ for each λ.

Restricting β to U_λ gives a section β_λ of κ_λ whose zero divisor is mD_λ, and a corresponding almost complex structure $J_\lambda = J_{\beta_\lambda}$ on U_λ. Then the image of any J_λ-holomorphic map lies in D_λ, so J_λ determines local invariants $GW_{g,n}^{loc}(D_\lambda, m, d)$ of U_λ for the class $d[D_\lambda]$ (with $d > 0$). Because β_λ and J_λ vary smoothly in λ, we then have

$$GW_{g,n}^{\mathrm{loc}}(D_\lambda, m, d) = GW_{g,n}^{\mathrm{loc}}(D_0, m, d)$$

for each λ. The righthand side of the above equals $L_{g,n}(N, m, d)$ by definition, while for $\lambda \neq 0$ the lefthand side is $GW_{g,n}^{loc}(D, m, d)$ because Z_λ is biholomorphic to X by a map that takes D_λ to D. This completes the proof of the lemma.   q.e.d.

Example 4.4. Let $\pi : E(m + 2) \to \mathbb{P}^1$ be an elliptic surface with $12(m+2)$ singular fibers which are all nodal. This surface is K3 if $m = 0$ and properly elliptic if $m > 0$. By the canonical divisor formula (see (6.1) below) the canonical bundle of $E(m + 2)$ is $\pi^* \mathcal{O}(m)$. Thus the generic canonical divisor is the sum of m disjoint regular fibers F_i, and for any regular fiber F the divisor mF is also a canonical divisor. Using Lemma 4.3 and equation (4.1), we then have

$$L_{g,n}(\mathcal{O}, m, d[F]) = GW_{g,n}(E(m + 2), d[F]) = m\, L_{g,n}(\mathcal{O}, d[F]).$$

5. Exceptional curves and blowups

This section establishes a "blowup formula" that reduces the problem of computing GW invariants to the case of minimal surfaces. This extends some previous partial blowup formulas, cited at the end of this

section. In our approach the blowup formula is a consequence of the
localization Lemma 3.2.

First consider a closed symplectic 4-manifold X with an almost com-
plex structure J and an exceptional J-holomorphic curve E. We can
then consider the (global) invariants

$$(5.1) \qquad\qquad GW_{g,n}(X, d[E]),$$

which give the contributions to GW_X of all maps whose image represents
a multiple of $[E]$. Fix a diffeomorphism $\iota : \mathbb{P}^1 \to E$ and let ι_* denote
the map $H_*(\overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^1)^n) \to H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$ induced by ι.

Lemma 5.1. *For $d > 0$, (5.1) is given by the local invariant of
Example* 4.2:
$$GW_{g,n}(X, d[E]) = \iota_* L_{g,n}(\mathcal{O}(-1), d).$$

Proof. Since $E^2 = -1$, any J-holomorphic curve representing a class
$d[E]$ has an image in E. Thus,
$$GW_{g,n}(X, d[E]) = GW_{g,n}^{\mathrm{loc}}(E, d).$$

After rescaling the symplectic form on $\mathbb{P}^1$ we may assume that $\iota : \mathbb{P}^1 \to$
E is a symplectomorphism. By the Symplectic Neighborhood Theorem
this extends to a symplectomorphism $\varphi : U \to V$ from a neighborhood
U of the zero section in $\mathcal{O}(-1) \to \mathbb{P}^1$ to a neighborhood V of E in X.
Pushing the standard complex structure J_0 on $\mathcal{O}(-1)$ forward by φ gives
an almost complex structure J_0' on V that makes φ an isomorphism of
almost complex neighborhoods. Furthermore, E is a J_0' holomorphic
curve, so the local invariant above can be calculated using J_0'. Thus,
when $d > 0$,
$$GW_{g,n}^{\mathrm{loc}}(E, d) = \iota_* L_{g,n}(\mathcal{O}(-1), d).$$

<div align="right">q.e.d.</div>

Let X be a compact Kähler surface with $p_g > 0$ and let $\pi : \tilde{X} \to$
X be the blowup of X at a point p. Different choices of the point p
yield surfaces $\tilde{X}$ that are symplectic deformation equivalent, so the GW
invariants of $\tilde{X}$ are independent of the choice of p. Note that every
$A \in H_2(\tilde{X})$ can be uniquely written as $A = B + dE$ where E is the class
of the exceptional curve and $B \cdot E = 0$ and the invariant $GW_{g,n}(X, \pi_* B)$
can be regarded as a homology class in $H_*(\overline{\mathcal{M}}_{g,n} \times (X \setminus \{p\})^n)$.

Proposition 5.2. *Let X be a compact Kähler surface with $p_g > 0$
and let $\pi : \tilde{X} \to X$ be its blowup at a point p. Then the GW invariant
of each class $A = B + dE$ as above is given by*

$$(5.2) \quad GW_{g,n}(\tilde{X}, A) = \begin{cases} L_{g,n}(\mathcal{O}(-1), d) & \text{if } A = dE \text{ with } d > 0 \\ \pi_*' GW_{g,n}(X, \pi_* A) & \text{if } A \cdot E = 0 \\ 0 & \text{otherwise} \end{cases}$$

where π'_* is the induced homology map by the composition of the isomor-phism $X \setminus \{p\} \to \tilde{X} \setminus E$ and the inclusion $\tilde{X} \setminus E \to \tilde{X}$.

Proof. Fix a holomorphic $(2,0)$ form α on $\tilde{X}$ with zero divisor $D \in |K_X|$ and a blowup point $p \notin D$. Then $\tilde{\alpha} = \pi^*\alpha$ is a holomorphic $(2,0)$ form on $\tilde{X}$ whose zero divisor $\tilde{D} \in |K_{\tilde{X}}|$ is the disjoint union of $\pi^*(D)$ and the exceptional curve E. Each class $A \in H_2(\tilde{X})$ with non-zero GW invariant can be represented by a $J_{\tilde{\alpha}}$-holomorphic map $f : C \to \tilde{X}$ from a connected curve C. By Lemma 3.2, the image of f lies in $\tilde{D}$. Hence either $A \cdot E = 0$ or $A = dE$ with $d > 0$. The case $A = dE$ was done in Lemma 5.1.

If $A \cdot E = 0$, choose a sequence of almost complex structures J_ℓ converging to $J_{\tilde{\alpha}}$. As $\ell \to \infty$, the J_ℓ-holomorphic maps converge pointwise to $J_{\tilde{\alpha}}$-holomorphic maps. These limit maps lie in $\tilde{D}$ but not in E because of the condition $A \cdot E = 0$. Thus for large ℓ the images are bounded away from E; in fact, they are uniformly bounded away from E for f in the compact space $\overline{\mathcal{M}}_{g,n}^{J_\ell}(\tilde{X}, A)$ of stable maps. Consequently, the condition that J_ℓ is generic for this space of stable curves is the same as the condition that an almost complex structure that agrees with $\pi_* J_\ell$ outside a sufficiently small neighborhood of the blowup point is generic for the corresponding space of stable maps into X. When both are generic, composition with π gives a diffeomorphism

$$\mathcal{M}_{g,n}^{J_\ell}(\tilde{X}, A) \xrightarrow{\cong} \mathcal{M}_{g,n}^{J_\ell}(X, \pi_* A)$$

that respects orientations and the stabilization and evaluation maps. Hence the corresponding GW invariants are equal.   q.e.d.

Remark 5.3. The hypothesis $p_g > 0$ is needed in Proposition 5.2. For example, when X is $\mathbb{P}^2$ and L is the class of the line, the invariants $GW_{g,n}(\tilde{X}, aL + bE)$ with $b > 1$ are non-zero: they are enumerative counts of the curves in $\mathbb{P}^2$ satisfying certain contact and tangency conditions at the blowup point (see Gathmann [7]). Jianxun Hu showed that the part of Proposition 5.2 pertaining to classes A with $A \cdot E = 0$ and $A \cdot E = 1$ hold on any symplectic manifold ([11]). For other classes, however, the contrast between Proposition 5.2 and Gathmann's results for $\mathbb{P}^2$ shows that any universal blowup formula for GW invariants must distinguish rational surfaces from those with $p_g > 0$.

The first part of the Structure Theorem 0.1 is a version of the blowup formula (5.2). Given a compact Kähler surface X with $p_g > 0$, let $\pi : X \to X'$ be the projection to the minimal model. By perturbing the blowup points, we can insure that there is a canonical divisor on X whose support is a disjoint union of exceptional curves $\{E_k\}$ and other curves D_ℓ. Define a formal power series with coefficients in $H_*(\overline{\mathcal{M}}_{g,n} \times$

$(\mathbb{P}^1)^n)$ by setting

$$(5.3) \qquad L^0(t) = \sum_{d>0} \sum_{g,n} \frac{1}{n!} L_{g,n}(\mathcal{O}(-1), d) \, t^d \, \lambda^{2g-2}$$

and another with coefficients in $H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$ by

$$(5.4) \qquad GW'_X = \sum_{A \neq 0} \sum_{g,n} \frac{1}{n!} \pi'_* GW_{g,n}(X, \pi_* A) \, t_A \, \lambda^{2g-2}.$$

The blowup formula then gives the following succinct equation (cf. Theorem 0.1).

Proposition 5.4. *The GW invariant of X is a sum*

$$GW_X = GW_X^0 + \sum_{E_i} L^0(t_{E_i}) + GW'_X.$$

6. The Structure Theorem for properly elliptic surfaces

In light of the blowup formula of the previous section, we can henceforth assume that all surfaces X are minimal. Furthermore, the GW invariants of a K3 or abelian surface are trivial by Corollary 3.3. The Enriques-Kodaira classification then shows that, among minimal surfaces with $p_g > 0$, there are two cases left to consider: minimal properly elliptic surfaces and minimal surfaces of general type. We will consider these separately.

Let $\pi : X \to C$ be a minimal properly elliptic surface. Then the sheaf $L = (R^1 \pi_* \mathcal{O}_X)^{-1}$ is a line bundle on C with $\deg L = \chi(\mathcal{O}_X) \geq 0$, and the canonical bundle is

$$K_X = \pi^*(L \otimes K_C) \ \otimes \ \mathcal{O}(\sum_k (m_k - 1)F'_k),$$

where F'_k are multiple fibers of multiplicity m_k ([**6**] pages 47-49). Correspondingly, each canonical divisor of X has the form

$$(6.1) \qquad \sum_j n_j F_j + \sum_k (m_k - 1)F'_k$$

where $\sum n_j F_j$ is the pullback of a divisor in $|L + K_C|$ of degree $k_\pi = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C)$. In general, the fibers F_j need not to be smooth or disjoint from the F'_k.

Proposition 6.1. *Every minimal properly elliptic surface $\pi : X \to C$ can be deformed to a minimal properly elliptic surface whose generic canonical divisor has the form (6.1) where the F_j and F'_k are disjoint smooth fibers.*

Proof. By a theorem of Moishezon X can be deformed to a minimal properly elliptic surface whose only singular fibers are reduced nodal curves and multiple fibers with smooth reduction (see [**6**] p. 113 and [**4**] p. 266). This deformed X is obtained by log transforms on an elliptic surface $\pi : S \to C$ without multiple fibers whose canonical bundle is $K_S = \pi^*(L + K_C)$ for the same line bundle L ([**6**] pp. 102–103). By deforming the fibers on which the logarithmic transformations are done, we can assume that none of the fibers F'_k lie over the base points of the linear system $|L + K_C|$, and hence the generic canonical divisor of X has the form (6.1) with $F_j \cap F'_k = \emptyset$ for all j and k. It therefore suffices to prove Proposition 6.1 for the surface S.

Next note that $|L+K_C|$ is empty when $\deg(L+K_C) = \deg L + 2g(C) - 2 \leq 0$ and has a base point at $p \in C$ if and only if $h^0(L + K_C - p) = h^0(L + K_C)$ (see [**10**] p. 308). By Riemann-Roch and Serre duality, this last condition is equivalent to $h^0(p - L) = h^0(-L) + 1$. Hence, $|L + K_C|$ has no base points when $\deg L \geq 2$, and also when $\deg L = 1$ and $L \neq \mathcal{O}(p)$ for any $p \in C$. In these cases Bertini's Theorem implies the generic canonical divisor is the disjoint union of distinct smooth fibers. This leaves only two specific cases:

   a) $\deg L = 0$ and $g = g(C) \geq 2$, and
   b) $L = \mathcal{O}(p)$ for some $p \in C$ and $g \geq 1$.

In fact, case a) occurs only when S has no singular fibers ([**6**], p. 48). Thus the proposition is true in case a).

In case b), choose points $p, q \in C$ that are not linearly equivalent, and let L be any one of the 2^{2g} line bundles on C with $L^2 = \mathcal{O}(p + q)$. Following [**6**] p. 60, one can construct an elliptic surface $\pi_L : S_L \to C$ with section with $(R^1\pi_{L*}\mathcal{O}_{S_L})^{-1} = L$ whose only singular fibers are the fibers over p and q. It follows from Seiler's Theorem (Corollary I.5.14 of [**6**]) that each S_L is deformation equivalent to S. Since L is not isomorphic to $\mathcal{O}(p)$ or $\mathcal{O}(q)$, the generic element of $|L + K_C|$ has support disjoint from p and q. The corresponding canonical divisor of S_L is then a union of smooth fibers.   q.e.d.

Remark 6.2. R. Friedman (private communication) has proved a stronger version of Proposition 6.1: one can assume, after further deformations, that $n_j = 1$ for all j. This is a more natural statement, but is not needed for our purposes in light of the calculation of Example 4.4.

Proposition 6.1 is useful because *Kähler surfaces that are deformation equivalent as complex surfaces have the same GW invariants.* This is true because deformation equivalent surfaces are smoothly isotopic ([**6**] page 18) and, because the space of Kähler forms with a fixed orientation is convex, that isotopy lifts to give a symplectic deformation equivalence. Consequently, the GW invariants are the same.

Thus we may assume that the generic canonical divisor D has the form (6.1) where

- each F_j is a regular fiber with holomorphically trivial normal bundle, and
- smooth multiple fiber F_{m_k} whose normal bundle N_k is torsion of order m_k in the group $\mathrm{Pic}^0(F_{m_k})$ of line bundles of degree zero

(cf. Section III.8 of [**4**]). Then for a regular fiber F_j with $n_j = 1$, we have the local GW invariants (4.3) with $m = 1$ and $N = \mathcal{O}$. These define a function L^1 as follows.

Definition 6.3. Let $\mathcal{O}$ is the trivial line bundle over T^2 and set

$$L^1(t) = \sum_{d>0} \sum_{g,n} \frac{1}{n!} L_{g,n}(\mathcal{O}, d) \, t^d \, \lambda^{2g-2}.$$

For a regular fiber F_j with $n_j > 1$, one can form the corresponding power series with $L_{g,n}(\mathcal{O}, d)$ replaced by $L_{g,n}(\mathcal{O}, n_j, d)$. The result is simply $n_j L^1(t)$ by the calculation of Example 4.4.

For multiple fibers, we will define similar functions $L^2_m(t)$ in terms of the GW invariants of a "model space" constructed by a logarithmic transformation. To that end, fix an elliptic $K3$ surface $X \to \mathbb{P}^1$, a regular F of X and a torsion line bundle $\xi \in Pic^0(F)$ of order $m > 1$. Applying the logarithmic transformation defined by this data yields an elliptic surface $X(F, \xi)$. This surface

- is simply connected and therefore Kähler (see [**8**] and Theorem 3.1 of [**4**]), and
- has $\chi(\mathcal{O}_X) = 2$, so by (6.1) its canonical divisor $D = (m-1)F'_m$ is supported on a single multiple fiber F'_m of multiplicity m.

Changing the choices of X, F and ξ yields a surface that is deformation equivalent to $X(F, \xi)$ (Theorem I.7.6 of [**6**]) and hence has the same GW invariants. We will write $K3(m)$ for the generic surface in this deformation class.

Definition 6.4. With $K3(m)$ and F'_m as above, set

$$L^2_m(t) = \sum_{d>0} \sum_{g,n} \frac{1}{n!} GW_{g,n}(K3(m), d[F'_m]) \, t^d \, \lambda^{2g-2}.$$

The following proposition shows that the local invariants at any smooth multiple fiber F_m of multiplicity m can be expressed in terms of GW invariants of $K3(m)$ that are encoded in the function $L^2_m(t)$.

Proposition 6.5. *Let X be a properly elliptic surface with a smooth multiple fiber F_m of multiplicity $m \geq 2$. Then*

$$GW^{\mathrm{loc}}_{g,n}(X, m-1, d[F_m]) = GW_{g,n}(K3(m), d[F'_m]).$$

Proof. Recall that there is a local model for a neighborhood U of F_m (cf. Prop. 6.2 of [**6**]). Specifically, there is a (smooth) elliptic fibration $\pi_0 : U_0 \to \Delta$ over a unit disk $\Delta \subset \mathbb{C}$ and a torsion line bundle ξ of order m on $\pi_0^{-1}(0)$ such that U is isomorphic, as an elliptic fibration, to the elliptic fibration obtained by performing the m-logarithmic transformation defined by ξ on the central fiber $\pi_0^{-1}(0)$. In particular, π_0 and ξ completely determine the m-spin curve (F_m, N_m), that is, determine the curve F_m and a normal bundle N_m satisfying $N_m^m = K_{F_m}$.

Furthermore, there is a holomorphic function h_0 on Δ satisfying $\operatorname{Im} h_0(s) > 0$ such that $\pi_0 : U_0 \to \Delta$ is the quotient $(\mathbb{C} \times \Delta)/(\mathbb{Z} \times \mathbb{Z}) \to \Delta$ with the action of $\mathbb{Z} \times \mathbb{Z}$ given by

$$(m, n)(z, s) = (z + m + nh_0(s), s)$$

(p. 202 of [**4**]). Now fix a normal neighborhood of a smooth fiber of $K3 \to \mathbb{P}^1$. One can then choose an isomorphic (smooth) elliptic fibration $\pi_1 : U_1 \to \Delta$ over the unit disk Δ under which the fixed smooth fiber of $K3$ corresponds to the central fiber $\pi_1^{-1}(0)$. As above, this fibration is determined by a holomorphic function h_1 on Δ with $\operatorname{Im} h_1(s) > 0$.

Since for each $t \in [0, 1]$ the function $h_t = (1 - t)h_0 + th_1$ is holomorphic on Δ and satisfies $\operatorname{Im} h_t(s) > 0$, using h_t one can obtain a family of elliptic fibrations $\pi_t : U_t \to \Delta$. Then, performing m-logarithmic transformation on each fiber $\pi_t^{-1}(0)$ using a family of line bundles ξ_t of order m on $\pi_t^{-1}(0)$ with $\xi_0 = \xi$ shows that the m-spin curves defined by $F_m \subset X$ and a multiple fiber $F'_m \subset K3(m)$ are deformation equivalent. Therefore, we have

$$GW_{g,n}^{\mathrm{loc}}(X, m - 1, d[F_m]) = GW_{g,n}^{\mathrm{loc}}(K3(m), m - 1, d[F'_m])$$
$$= GW_{g,n}(K3(m), d[F'_m]),$$

where the first equality follows from Lemmas 4.1 and 4.3, and the second follows from (3.5) and the fact that the canonical divisor of $K3(m)$ is $(m - 1)F'_m$.   q.e.d.

The structure theorem for minimal properly elliptic surfaces follows immediately from (4.1), Lemmas 4.1 and 4.3, and Proposition 6.5. The result is the following case of Theorem 0.1.

Theorem 6.6. *If X is a minimal properly elliptic surface whose canonical divisor D is given as in* (6.1), *then*

$$GW_X = GW_X^0 + k_\pi L^1(t_F) + \sum_k L_{m_k}^2(t_{F_k}),$$

where F is a regular fiber and $t_{F_k}^{m_k} = t_F$.

7. The Structure Theorem for surfaces of general type

When X is a minimal surface of general type, every canonical divisor is connected and has arithmetic genus $h \geq 2$ ([**4**]). Unlike the case of elliptic surfaces, it is not always possible to deform a surface of general type to insure the existence of a smooth canonical divisor. For example, Bauer and Catanese have shown that there is a surface S with $p_g = 4$, $K^2 = 45$ that has no complex deformations and such that each canonical divisor is singular and reducible ([**3**]). It is not presently understood how common such examples are. To avoid this complication we make the following assumption.

Assumption. For some Kähler structure in the deformation class of X, there is a smooth canonical divisor D with multiplicity 1.

(Of course, if this is true for some Kähler structure then it is true for the generic one.) When D is smooth with multiplicity 1, the adjunction formula shows that the normal bundle N of D is a holomorphic square root of K_D:

$$(7.1) \qquad\qquad\qquad N^2 = K_D.$$

Recall that a *theta characteristic* on a smooth curve D is a line bundle N with $N^2 = K_D$. In the special case when $K_D = \mathcal{O}$ is trivial, the set $S(D)$ of all theta characteristics is the same as the group $J_2(D)$ of points of order 2 in the Jacobian. In general, $S(D)$ is a principal homogeneous space for $J_2(D)$ with the obvious action: if N is a theta characteristic and $L^2 = \mathcal{O}$, then $N \otimes L$ is another theta characteristic. Since $J_2(D)$ is naturally isomorphic to $H^1(D; \mathbb{Z}_2)$, there are 2^{2h} theta characteristics on a curve of genus h. A theta characteristic N is *even* or *odd* according to the parity of $h^0(D, N)$.

A *spin curve* is a pair (D, N) consisting of a curve with a theta characteristic. The spaces $\mathcal{S}_{h,+}$ (resp. $\mathcal{S}_{h,-}$) of all genus h even (resp. odd) spin curves have compactifications $\overline{\mathcal{S}}_{h,\pm}$. The following three facts are classical.

Proposition 7.1 (see [**1**], [**2**], and [**5**]). *Let D be a smooth curve of genus h.*

  (a) *There are $2^{h-1}(2^h + 1)$ even and $2^{h-1}(2^h - 1)$ odd theta characteristics.*
  (b) *$h^0(D_t, N_t) \bmod 2$ is constant along any smooth family (D_t, N_t) of spin curves.*
  (c) *$\overline{\mathcal{S}}_{h,\pm}$ is an irreducible projective variety and $\partial \mathcal{S}_{h,\pm} = \overline{\mathcal{S}}_{h,\pm} \setminus \mathcal{S}_{h,\pm}$ is a proper analytic subvariety.*

Corollary 7.2. *The invariants $L_{h,n}(N, d)$, defined by (4.3) when $m = 1$, depend only on the genus h and the parity of $h^0(D, N)$.*

Proof. Since $\overline{\mathcal{S}}_{h,\pm}$ is irreducible, the smooth part $\overline{\mathcal{S}}_{h,\pm}^*$ is connected ([**9**] page 21), and hence $\overline{\mathcal{S}}_{h,\pm}^* \setminus \partial\mathcal{S}_{h,\pm}$ is connected. Thus any two smooth spin curves of the same parity can be joined by a path of spin curves. The corollary then follows from Lemma 4.1.   q.e.d.

For our case — a surface of general type with a smooth canonical divisor D with multiplicity 1 — the parity of $h^0(D, N)$ is actually a *global* invariant, as the following lemma shows.

Lemma 7.3. *If X is a minimal surface of general type and $D \subset X$ is a smooth canonical divisor with normal bundle N, then*

$$h^0(D, N) \equiv \chi(\mathcal{O}_X) \ (\mathrm{mod}\ 2).$$

Proof. Since N is the restriction of K to D, there is an exact sequence $0 \to \mathcal{O}_X \overset{m}{\to} \mathcal{O}_X(K) \overset{r}{\to} \mathcal{O}_D(N) \to 0$, where $m(f) = f\alpha$ and $r(\beta) = \beta_{|C}$. This induces a long exact sequence of cohomology which, using the isomorphisms $H^{0,1}(X) \cong H^1(\mathcal{O}_X)$ and $H^1(K) \cong H^{2,1}(X)$, begins

$$0 \to H^0(\mathcal{O}_X) \to H^0(K) \to H^0(N) \to H^{0,1}(X) \overset{m_*}{\to} H^{2,1}(X) \to \cdots$$

where m_* is given by $m_*(\lambda) = \lambda \wedge \alpha$. The hermitian inner product on $H^{0,1}(X)$ gives an orthogonal splitting $H^{0,1}(X) = \ker m_* \oplus V$ and, by the above sequence, $h^0(N) = p_g + q - 1 - \dim V$. Since $\chi(\mathcal{O}_X) = 1 - q + p_g$, it suffices to show that V is even dimensional. After composing with the star operator, $L = *m_* : H^{0,1}(X) \to H^{0,1}(X)$ satisfies

$$(7.2) \qquad\qquad \langle\, \lambda\, , L(\delta)\,\rangle = -\langle\, \delta\, , L(\lambda)\,\rangle.$$

Thus, L induces a nondegenerate sympletic pairing on $H^{0,1}(X)/\ker m_*$ $\cong V$, so $\dim V$ is even.   q.e.d.

We can proceed as we did for elliptic surfaces. Again, we first define invariants associated with a spin curve.

Definition 7.4. For each smooth genus $h \geq 2$ spin curve (D, N), let $L_{g,n}(N, d)$ be the local GW invariant (4.3) and set

$$(7.3) \qquad L_{h,\pm}^3(t) = \sum_{g,n} \sum_{d \geq 1} \frac{1}{n!}\, L_{g,n}(N, d)\, t^d \lambda^{2g-2}.$$

This notation incorporates the fact that, by Corollary 7.2, this series depends only on h and the parity of (D, N).

For minimal surfaces of general type, the statement of the structure theorem is especially simple because the canonical divisor of X has a single component. The GW series is obtained from one of the series (7.3).

Theorem 7.5. *Suppose that X is a minimal surface of general type with a smooth, multiplicity 1 canonical divisor D. Let $h = K_X^2 + 1$ be the genus of D. Then (again suppressing inclusion maps)*

$$GW_X = GW_X^0 + \begin{cases} L_{h,+}^3(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is even} \\ L_{h,-}^3(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is odd.} \end{cases}$$

Consequently, the GW series of X depends only on h and $\chi(\mathcal{O}_X)$.

Proof. This follows directly from (4.1), Lemma 4.3, Corollary 7.2 and Lemma 7.3.   q.e.d.

8. Moduli spaces and linearizations

For each fixed $\alpha \in \mathcal{H}$, we can consider the linearization D_f of the J_α-holomorphic map equation at each J_α-holomorphic map $f : C \to X$. This operator is important for local descriptions of the moduli space. After a brief discussion of moduli spaces, we will write down the formula for D_f and show that it has some remarkable analytic properties.

Consider a smooth component D of a canonical divisor of X. When D has multiplicity 1, we have $N^2 = K_D$ as in (7.1). When $D = F_m$ is a multiple elliptic fiber with multiplicity m, the normal bundle satisfies $N^m = \mathcal{O}_D$. Taking Chern classes, both cases give the formula

$$(8.1) \qquad\qquad c_1(N)[D] = h - 1.$$

Lemma 8.1. *Fix a smooth genus h component $D \subset X$ of a canonical divisor. Then the (formal real) dimensions of the moduli spaces $\mathcal{M}_g(D, d)$ (of degree d genus g covers of the curve D) and $\mathcal{M}_g(X, d[D])$ (of maps from a genus g curve representing $d[D] \in H_2(X)$) are*

$$(8.2) \qquad \dim \mathcal{M}_g(D, d) = 4\beta \qquad and \qquad \dim \mathcal{M}_g(X, d[D]) = 2\beta$$

where $\beta = d(1 - h) + g - 1$.

Proof. The restriction of TX to D decomposes as $TD \oplus N$. Using (8.1) we then have $K_X \cdot D = K_D \cdot D - c_1(N)[D] = h - 1$. Both parts of (8.2) then follow from the dimension formula (1.1).   q.e.d.

To interpret the number β geometrically, consider a J_α-holomorphic map $f : C \to D$ from a smooth genus g curve onto D. The canonical classes of C and D are then related by the Riemann-Hurwitz formula $K_C = f^* K_D + B$ where B is the ramification divisor. Consequently, the number of branch points, counted with multiplicity, is

$$(8.3) \qquad |B| = 2\beta \qquad \text{where } \beta = d(1 - h) + g - 1.$$

To proceed, we need explicit formulas. By a standard calculation (cf. [15], [21]), the linearization of the J_α-holomorphic map equation,

evaluated at a map f and applied to a variation ξ of the map and a variation k of the complex structure on the domain, is

$$(8.4) \qquad D_f(\xi, k) = L_f(\xi) + J_\alpha df k$$

where the operator $L_f : \Omega^0(f^*TX) \to \Omega^{0,1}(f^*TX)$ is given by

$$L_f(\xi)(w) = \bar{\partial}_f \xi(w) + \left(\frac{1}{2} J \nabla_\xi J + \nabla_\xi K_\alpha \right) (df jw) + K_\alpha(\nabla \xi) jw$$

for each $w \in \Omega^0(TC)$ (here $\bar{\partial}_f \xi(w)$ is $\frac{1}{2} (\nabla_w \xi + J\nabla_{jw}\xi)$). In our case $\nabla J = 0$ and α vanishes along the image of f, so that

$$(8.5) \qquad L_f = \bar{\partial}_f + R_\alpha$$

with

$$(8.6) \qquad R_\alpha(\xi) = -(\nabla_\xi K_\alpha) \circ df \circ j.$$

Lemma 8.2. *Let D be a smooth component of a canonical divisor D_α and N be the normal bundle of D. Then, for each $p \in D$, $u \in T_p D$ and $\xi \in N_p$ we have*

(a) $\nabla_u K_\alpha = 0$,   (c) $\nabla_{J\xi} K_\alpha(u) = -J\nabla_\xi K_\alpha(u)$,

(b) $\nabla_\xi K_\alpha(u)$ *is orthogonal to* $T_p D$,   (d) $|\nabla_\xi K_\alpha(u)|^2 = |\nabla\alpha|^2 |\xi|^2 |u|^2$.

Proof. Since $\alpha \equiv 0$ along D, (a) follows from Lemma 2.1a. Next, the fact α is a closed 2-form and $\nabla_u \alpha = 0$ gives the formula

$$0 = d\alpha(u, \xi, \eta) = (\nabla_\xi \alpha)(\eta, u) - (\nabla_\eta \alpha)(\xi, u)$$

for any $\eta \in T_p X$. Applying the definition of K_α, this becomes

$$(8.7) \qquad \langle \eta, \nabla_\xi K_\alpha(u) \rangle = \langle \xi, \nabla_\eta K_\alpha(u) \rangle.$$

When $\eta \in T_p D$, we have $\nabla_\eta K_\alpha = 0$ and thus (8.7) shows (b). Then, because J is skew-adjoint with $\nabla J = 0$, and $K_\alpha J = -JK_\alpha$, (8.7) implies that

$$\langle \eta, \nabla_{J\xi} K_\alpha(u) \rangle = \langle J\xi, \nabla_\eta K_\alpha u \rangle = \langle \xi, \nabla_\eta K_\alpha(Ju) \rangle$$
$$= \langle \eta, \nabla_\xi K_\alpha(Ju) \rangle = -\langle \eta, J\nabla_\xi K_\alpha(u) \rangle.$$

This gives (c). Finally, noting $\nabla K_\alpha(Ju) = -J\nabla K_a(u)$ and using (c), the fact TD and N are J-invariant and Lemma 2.1, we have

$$|\nabla_\xi K_\alpha(u)|^2 = |\nabla K_\alpha|^2 |\xi|^2 |u|^2 = |\nabla\alpha|^2 |\xi|^2 |u|^2.$$

$$\text{q.e.d.}$$

As an immediate corollary, we have:

Corollary 8.3. *If $f : C \to D$ is a J_α-holomorphic map onto a smooth component D of a canonical divisor D_α, then R_α vanishes on f^*TD and defines a (real) bundle map*

$$R_\alpha : f^*N \to T^{0,1}C \otimes f^*N,$$

where N is the normal bundle to D. This R_α satisfies $R_\alpha J = -J R_\alpha$ and

$$(8.8) \qquad\qquad |R_\alpha(\xi)|^2 = |\nabla\alpha|^2 \, |\xi|^2 \, |df|^2.$$

On a Kähler surface X, each $\alpha \in \mathcal{H}$ has an associated almost complex structure J_α and canonical divisor D_α. Let V is a smooth component of the support of D_α. Following [**13**], one can use the space of (J_α, ν)-holomorphic maps to define the relative GW invariant for the pair (X, V) provided (J_α, ν) is a generic "V-compatible" pair as defined in Section 3 of [**13**].

Corollary 8.4. *Let V be a smooth component of the support of D_α. If $\nabla\alpha \equiv 0$ on V, then $(J_a, 0)$ is a V-compatible pair, while if $\nabla\alpha \not\equiv 0$ then (J_α, ν) is V-compatible for no choice of ν.*

Proof. Let π_N denote the orthogonal projection onto the normal bundle N of V. A pair (J_α, ν) is V-compatible if it satisfies three conditions: J_α preserves TV, ∇J_α satisfies

$$(8.9) \qquad \pi_N\left[(\nabla_\xi J_\alpha + J_\alpha \nabla_{J_\alpha \xi} J_\alpha)(u)\right] = \pi_N\left[(\nabla_u J_\alpha + J_\alpha \nabla_{J_\alpha u} J_\alpha)(\xi)\right]$$

for all $u \in TD$ and $\xi \in N$, and ν and $\nabla\nu$ satisfy conditions that are automatically true when $\nu = 0$. Since $\alpha = 0$ along V, the definition (2.3) of J_α shows that $J_\alpha = J$ and $\nabla J_\alpha = -2\nabla K_\alpha$ at each point in V. Thus V is J_α-holomorphic. One can then use Lemma 8.2 to see that Condition (8.9) is equivalent to

$$\nabla_\xi K_\alpha(u) = 0 \quad \forall u \in TD, \ \forall \xi \in N.$$

Lemma 8.2 d then implies that V-compatibility conditions hold only if $\nabla\alpha = 0$ along V, and that if $\nabla\alpha = 0$ along V then $(J_a, 0)$ satisfies the V-compatibility conditions. q.e.d.

The two terms of the operator (8.5) satisfy a remarkable property under the L^2 pairing:

Lemma 8.5. *Let D be a smooth component of a canonical divisor D_α with normal bundle N. Then for each J_α-holomorphic map $f : C \to D$ we have*

$$\int_C \langle \bar\partial\xi, \ R_\alpha\eta \rangle + \int_C \langle \bar\partial\eta, \ R_\alpha\xi \rangle = 0 \quad \forall \xi, \eta \in \Omega^0(f^*N).$$

Proof. Let $f_{s,t}$ be a 2-parameter family of deformations of the map $f = f_{0,0}$ with $\frac{d}{ds}f|_{s=t=0} = \xi$ and $\frac{d}{dt}f|_{s=t=0} = \eta$. Then $\bar{\partial}f_{s,t} = \bar{\partial}(s\xi + t\eta) + Q(s,t)$ where Q is at least quadratic in (s,t). Since the image of f represents a multiple of the (1,1) class $[D]$, equation (3.1) gives

$$0 = \int f^*\alpha = \int \langle \bar{\partial}f, \; K_\alpha(df j) \rangle$$

for each $f = f_{s,t}$. Now differentiate this equation with respect to both s and t and evaluate at $s = t = 0$, noting that $\bar{\partial}f$ and $K_\alpha(df j)$ both vanish at $s = t = 0$. The result is

$$0 = \int \langle \bar{\partial}\xi, \; \nabla_\eta K_\alpha(df j) \rangle + \int \langle \bar{\partial}\eta, \; \nabla_\xi K_\alpha(df j) \rangle.$$

The lemma follows by the definition of R_α. q.e.d.

We finish this section by discussing the operator given by the normal component of the linearization (8.5). For each map $f : C \to D$ as in Corollary 8.3 the pullback f^*TX of the tangent bundle decomposes orthogonally as $f^*TX = f^*TD \oplus f^*N$. Let π^N be the projection onto f^*N. The normal component $\pi^N \circ \nabla$ of the connection on f^*TX is a hermitian connection on f^*N; its (0,1) part defines an operator $\bar{\partial}_f^N$ and hence a holomorphic structure on f^*N. The restriction of $\bar{\partial}_f$ to f^*N then has the form

$$\bar{\partial}_f|_{f^*N} = \bar{\partial}_f^N + A$$

where A is a bundle map $f^*N \to T^{0,1}C \otimes f^*TD$ (which vanishes if f^*N is a holomorphic subbundle; see [**9**] pg. 78). On the other hand, since f^*TD is a holomorphic subbundle, the restriction of $\bar{\partial}_f$ to f^*TD is an operator $\bar{\partial}_f^T$ on f^*TD which is the usual $\bar{\partial}$-operator. Corollary 8.3 then implies that the linearization (8.4), as an operator

$$D_f : \Omega^0(f^*TD \oplus f^*N) \oplus H^{0,1}(TC) \to \Omega^{0,1}(f^*TD \oplus f^*N),$$

is given by

$$(8.10) \qquad D_f = \begin{pmatrix} \bar{\partial}_f^T & A \\ 0 & L_f^N \end{pmatrix} \oplus Jdf$$

where $L_f^N = \bar{\partial}_f^N + R_\alpha$. The next result shows that L_f^N is injective.

Proposition 8.6. *Suppose that $f : C \to D$ is a J_α-holomorphic map from a smooth curve onto a smooth component D of a canonical divisor D_α, and that either* (i) $\nabla\alpha \neq 0$ *somewhere on D, or* (ii) $\ker \bar{\partial}_f^N = 0$. *Then*

$$(8.11) \qquad \ker L_f^N = 0.$$

Proof. Suppose there is a non-zero $\xi \in \ker L_f^N$. Then the integral

$$\|L_f^N \xi\|^2 = \|(\overline{\partial}_f^N + R_\alpha)\xi\|^2 = \int_C |\overline{\partial}_f^N \xi|^2 + |\nabla\alpha|^2 \, |\xi|^2 \, |df|^2$$

vanishes (here we have used (8.8) and noted that, because $R_\alpha\xi$ is normal and $A\xi$ is tangent, Lemma 8.5 holds with $\overline{\partial}_f\xi = \overline{\partial}_f^N \xi + A\xi$ replaced by $\overline{\partial}_f^N \xi$). But both ξ and f satisfy elliptic equations, so by the Unique Continuation Theorem for elliptic equations $|\xi|^2 \, |df|^2$ is not zero on any open set. We conclude that $\overline{\partial}_f^N \xi = 0$ and $\nabla\alpha \equiv 0$ along D.   q.e.d.

9. Zero-dimensional spaces of stable maps

The simplest GW invariants are those associated with a space of stable maps whose formal dimension is zero. Such stable maps are especially simple: Lemma 9.1 below shows that they are unramified maps from smooth domains, and that the linearization D_f is invertible. Thus all zero-dimensional GW invariants are signed counts of the number of connected etale covers. This section establishes some basic facts needed to make these counts. Specific computations are done in Section 10.

The formal dimension of a space $\mathcal{M}_{g,n}(X, A)$ of stable maps is the index of linearization D_f at each $f \in \mathcal{M}_{g,n}(X, A)$. Calculating as in the proof of Lemma 8.1, one finds that index $D_f = 2\beta + 2n$ and similarly index $L_f^N = -2\beta + 2n$. Consequently, when the space of stable maps is formally 0-dimensional and the domain curve is smooth, we have

(9.1)   index D_f = index $L_f^N = 0$.

Now fix a smooth canonical divisor satisfying the conditions of Proposition 8.6. The Image Localization Lemma 3.2 implies that all invariants $GW_{g,n}(X, A)$ vanish unless A is a multiple $d[D]$ of the class of a component D of that canonical divisor. These invariants also vanish whenever the formal dimension of $\overline{\mathcal{M}}_{g,0}(X, A)$ is negative because the space $\overline{\mathcal{M}}_{g,0}(X, A)$, and therefore $\overline{\mathcal{M}}_{g,n}(X, A)$, is then empty for generic (J, ν). Thus, using the dimension formula of Lemma 8.1, we may assume that $A = d[D]$ and

(9.2) $\beta = n = 0$ with $\beta = d(1 - h) + g - 1$,

where h is the genus of D.

Lemma 9.1. *Suppose that $D \subset X$ is a smooth component of a canonical divisor D_α. Then any non-constant stable map $f : C \to D$ satisfying (9.2) is an etale cover from a smooth curve C and the linearization D_f is invertible.*

Proof. By (9.2) we have $g = dh - d + 1$. Suppose that C has ℓ irreducible components $\{C_i\}$. Restricting f to each component and

lifting to the normalization gives maps $\tilde{f}_i : \tilde{C}_i \to D$. Suppose that exactly k of these have degree $[\tilde{f}_i] = d_i > 0$. Then $\sum d_i = d$ and (8.3), applied to each C_i, gives

$$g = dh - d + 1 \leq \sum (d_i h - d_i + 1 + \beta_i) = \sum g_i' \leq \sum g_i \leq g$$

where β_i is the ramification index of $\tilde{f}_i$, g_i' is the geometric genus of C_i, and g_i is the arithmetic genus of C_i. This shows that $k = 1$ and C_1 has the same geometric and arithmetic genus. Consequently, C_1 is smooth of genus g and the remaining $\ell - k$ components have genus 0. Stability then implies that $\ell - k = 0$. Thus C is smooth and $f : C \to D$ has no critical points.

Recall that the linearization D_f is given by (8.10). The normal operator L_f^N is injective by (8.11) and hence is surjective by (9.1). Furthermore, Jdf induces an isomorphism from $H^{0,1}(TC)$ to $H^{0,1}(f^*TD) = \mathrm{coker}\,\bar{\partial}_f^T$, and therefore

$$\bar{\partial}_f^T \oplus Jdf : \Omega^0(f^*TD) \oplus H^{0,1}(TC) \to \Omega^{0,1}(f^*TD)$$

is also onto. Thus D_f is surjective with index zero, so is an isomorphism between the appropriate Sobolev spaces.   q.e.d.

When D_f is invertible, there is an associated invariant: its mod 2 spectral flow. That spectral flow is computed in the next proposition. This calculation is crucial to the discussion in the next section.

The mod 2 spectral flow of D_f is determined by choosing a path D_t of first order elliptic operators from an invertible complex linear operator D_0 to $D_1 = D_f$ so that D_t is invertible except at finitely many t_i along the way, and taking

(9.3) $$SF(D_f) = \sum_i \dim \ker D_{t_i} \qquad (\mathrm{mod}\ 2).$$

This is a homotopy invariant of the path, and is independent of D_0 because any two choices of D_0 can be connected by a path D_t of complex linear first order elliptic operators, and at each point along such a path $\ker D_t$ is even-dimensional.

Proposition 9.2. *Under the conditions of Lemma* 9.1,

(9.4) $$SF(D_f) \equiv h^0(f^*N) \quad (\mathrm{mod}\ 2).$$

Proof. First deform D_f to a diagonal operator along the path

$$D_t = \begin{pmatrix} \bar{\partial}_f^T & tA \\ 0 & L_f^N \end{pmatrix} \oplus Jdf.$$

Because both $\overline{\partial}_f^T \oplus Jdf$ and L_f^N are surjective, each D_t is surjective with index zero, so $\ker D_t = 0$ for all t. Noting that $\overline{\partial}_f^T \oplus Jdf$ is complex-linear, we then have

$$SF(D_f) = SF(D_0) = SF(L_f^N).$$

Next, since L_f^N is invertible by Lemma 8.6, $SF(L_f^N) = SF(L_f^N + B)$ for any sufficiently small compact perturbation B. Now write L_f^N as $\overline{\partial} + R_\alpha$ with $\overline{\partial} = \overline{\partial}_f^N$. Because index $\overline{\partial} = 0$ for etale covers, we can choose a complex-linear isomorphism $\bar{B} : \ker \overline{\partial} \to \operatorname{coker} \overline{\partial}$ and set $B = \bar{B}P$ where P is the L^2 orthogonal projection onto $\ker \overline{\partial}$. Then

$$D_t = \overline{\partial} + \delta B + tR_\alpha$$

is a path from $D_0 = \overline{\partial} + \delta B$ to $D_1 = L_f^N + \delta B$. Using Lemma 8.5, we have

$$\int |D_t \xi|^2 = \int |\overline{\partial}\xi|^2 + |(\delta B + tR_\alpha)\xi|^2.$$

This shows that D_0 is invertible and that $\ker D_t$ lies in $\ker \overline{\partial}$ and in $\ker(\delta \bar{B} + tR_\alpha)$ for each t. Taking δ sufficiently small, we then have

$$SF(L_f^N) = SF(D_1) = SF(\delta \bar{B} + \bar{R}_\alpha)$$

where $\bar{R}_\alpha$ is the restriction of R_α to $\ker \overline{\partial}$. But $\bar{R}_\alpha$ is injective and anti-commutes with J by Lemma 8.3. Furthermore, its image is L^2 perpendicular to the image of $\overline{\partial}$ by Lemma 8.5 and index $\overline{\partial} = 0$, so $\bar{R}_\alpha : \ker \overline{\partial} \to \operatorname{coker} \overline{\partial}$ is an isomorphism. This means that $SF(\delta \bar{B} + \bar{R}_\alpha)$ is the same as $SF(\bar{R}_\alpha)$ and, from the definition (9.3), the same as $SF(\bar{B}^{-1}\bar{R}_\alpha)$. Here $\bar{B}^{-1}\bar{R}_\alpha$ is an isomorphism of $H^0(C, f^*N)$ that anti-commutes with J. The lemma is completed using two simple facts about the spectral flow of finite-dimensional matrices:

  (a) $(-1)^{SF(A)} = \operatorname{sign} \det A$ for all $A \in GL(n, \mathbb{R})$.
  (b) If $A \in GL(2n, \mathbb{R})$ satisfies $JA = -AJ$ then $SF(A) = n \bmod 2$.

To see (a), choose a path A_t in the space of $n \times n$ matrices from A to Id; for a generic such path each kernel in (9.3) is 1-dimensional, so the spectral flow is the number of sign changes in $\det A_t$. For (b), choose a basis $\{v_1, Jv_1, \ldots, v_n, Jv_n\}$ and set $w_i = Av_i$. Then $v_1 \wedge Jv_1 \wedge \cdots \wedge Jv_n$ and $w_1 \wedge Jw_1 \wedge \cdots \wedge Jw_n$ both represent the complex orientation, so the calculation

$$\begin{aligned}
\det A \cdot v_1 \wedge JV_1 \wedge \cdots \wedge Jv_n &= Av_1 \wedge AJv_1 \wedge \cdots \wedge AJv_n \\
&= (-1)^n w_1 \wedge Jw_1 \wedge \cdots \wedge Jw_n
\end{aligned}$$

shows that $\operatorname{sign} \det A = (-1)^n$.   q.e.d.

In Gromov-Witten theory, the GW invariant associated with a zero-dimensional space of stable maps is the signed count of the maps in

that space with the sign of each map f specified by the mod 2 spectral flow of the linearization D_f (provided each D_f is an isomorphism). By Proposition 9.2 this sign is

$$(9.5) \qquad\qquad (-1)^{SF(D_f)} = (-1)^{h^0(f^*N)}.$$

This sign is well-defined even though $h^0(f^*N)$ may change under deformations of the holomorphic structure on f^*N. This is because, for etale covers $f : C \to D$, we have $f^*K_D = K_C$ and hence the equation $N^2 = K_{D_0}$ pulls back to $(f^*N)^2 = K_C$. Thus (C, f^*N) is a spin curve, so by Lemma 7.1 the parity of $h^0(f^*N)$ does not change as (D, N) is deformed.

Formula (9.5) is a key difference between GW invariants in two and four dimensions. The finite set of etale covers of D contribute to both the Gromov-Witten invariants of the curve D, and to the GW invariants of X through the inclusion $D \subset X$. But in the first case, each etale cover contributes $+1/|\mathrm{Aut}(f)|$ to the invariant, while in the second case the signs vary according to (9.5).

10. Zero-dimensional GW invariants: computations

The facts established in the previous section are enough to compute the contributions of etale covers to the GW series in some cases. We do this for the canonical class itself, for double covers, and for general etale covers for elliptic fibers.

The canonical class.

When X and D are as in Lemma 7.3, D is an embedded genus $g = K^2 + 1$ curve representing the canonical class K. For that genus, the GW invariant has dimension 0 by (1.1) and is immediately computable using (4.1), Proposition 9.2, and Lemma 7.3:

$$GW_g(X, K) = GW_g^{\mathrm{loc}}(D, 1) = (-1)^{h^0(N)} = (-1)^{\chi(\mathcal{O}_X)}.$$

This fact is well-known from other perspectives. In the context of Taubes' Gr invariant (see [**23**]), $g = K^2 + 1$ is the "embedded genus" case. In that case the Gr invariant is the same as the Seiberg-Witten invariant and is given by $Gr(K) = SW(K) = (-1)^{\chi(\mathcal{O}_X)}$. On the other hand, because D is embedded and connected, we also have $Gr(K) = GW_g(X, K)$.

Double covers.

The etale double covers of a curve D are classified by either $H^1(D; \mathbb{Z}_2)$ or, equivalently, by $J_2(D)$. In fact, if the square of a line bundle L is trivial, then L has a bisection s satisfying $s^2 = 1$ and the image of s is a smooth unramified double covering $f : C_L \to D$ that is connected

whenever $L \neq \mathcal{O}_D$. Such double coverings satisfy $f_* \mathcal{O}_{C_L} = \mathcal{O}_D \oplus L^{-1}$, and thus for any line bundle N on D

(10.1)
$$h^0(C_L, f^* N) = h^0(D, f_* f^* N) = h^0(D, N \otimes f_* \mathcal{O}_{\tilde{C}_L})$$
$$= h^0(D, N) + h^0(D, NL^{-1}).$$

Now suppose that D is a smooth component of a canonical divisor of genus h with normal bundle satisfying $N^2 = K_D$. Since each map f in the moduli space $\overline{\mathcal{M}}(D, 2)$ of etale double covers with connected domains has automorphism group $\mathbb{Z}_2$, each contributes $\pm \frac{1}{2}$ to the GW invariant, with the sign given by Proposition 9.2. Thus Proposition 7.1a, Lemma 7.3 and equation (10.1) yield

(10.2)
$$GW_g^{loc}(D, 2) = \sum_{f \in \overline{\mathcal{M}}(D, 2)} \frac{1}{2} (-1)^{h^0(f^* N)}$$
$$= \frac{1}{2} \left[(-1)^{h^0(N)} 2^h - 1 \right],$$

where $g = 2h - 1$. For surfaces of general type the sign $(-1)^{h^0(N)}$ can be calculated from the global invariant $\chi(\mathcal{O}_X) = 1 - q + p_g$ by Lemma 7.3.

Example 10.1. Exceptional curves have no etale double covers, while elliptic fibers have three connected double covers, all etale with genus 1. Thus (10.2) gives

1) A regular fiber F has trivial normal bundle, so $GW_1^{loc}(F, 2) = -\frac{3}{2}$.
2) A multiple fiber F_2 of order 2 has $h^0(N) = 0$, so $GW_1^{loc}(F_2, 2) = \frac{1}{2}$.
3) Formula (10.2) does not apply to multiple fiber F_m with multiplicity $m > 2$ because the normal bundle to F_m is not a theta characteristic, but instead satisfies $N^m = \mathcal{O}$. Nevertheless, we have $h^0(f^* N) = 0$ for each of the three nontrivial double covers f of F_m, so
$$GW_1^{loc}(F_m, m - 1, 2) = \frac{3}{2}.$$
4) When D is a smooth multiplicity 1 canonical divisor in a surface of general type, D has genus $h = K^2 + 1$ and a connected double cover $C \to D$ is etale if and only if C has genus $g = 2K^2 + 1$. By (10.2) the genus $g = 2h - 1$ invariant is
$$GW_g(X, 2K) = GW_g^{loc}(D, 2) = \frac{1}{2} \left[(-1)^{\chi(\mathcal{O}_X)} 2^h - 1 \right].$$

Etale Covers of Elliptic Fibers.

When (X, J) is a generic complex structure on a minimal properly elliptic surface, the generic canonical divisor has components of two types: smooth elliptic fibers and multiple fibers with smooth reduction. The simplest cases are regular fibers and multiple fibers of multiplicity

two. For those, we can give explicit formulas for the contributions to the GW invariants of smooth etale covers.

Regular Fibers. Every holomorphic map $f : C \to F$ onto a regular elliptic fiber has $f^*N = \mathcal{O}$, so $h^0(f^*N) = 1$. Such a map f is an etale cover if and only if C has genus $g = 1$. The stable moduli space $\overline{\mathcal{M}}_{1,0}(F, d)$ consists of $\sigma(d)$ points, where $\sigma(d) = \sum_{k|d} k$ is the sum of the divisors of d. Each of these is a generic as J_α-holomorphic map (Lemma 9.1 implies that $\ker D_f = 0$) with automorphism group of order d, and each is counted with a minus sign by Lemma 9.2 because $f^*N = O_C$. Thus the contribution of the etale covers to the local GW invariant of F is

$$\sum_{d>0} GW_1^{\mathrm{loc}}(F, d) \, t_F^d = -\sum_{d>0} \frac{\sigma(d)}{d} \, t_F^d = -\int \frac{G(t_F)}{t_F} \, dt_F,$$

where

$$G(t) = \sum \sigma(d) \, t^d = \prod_{k>0} \frac{k t^k}{1 - t^k}.$$

F_2 **Fibers.** As in (6.1), every elliptic fiber F_2 of multiplicity 2 is a component of each canonical divisor D_α with multiplicity 1. In particular, $\nabla \alpha$ does not vanish identically along F_2. Thus, by Lemma 9.1, $\ker D_f = 0$ and $\mathrm{coker}\, D_f = 0$ for every J_α-holomorphic etale cover $f : C \to F_2$. Consequently, the (local) GW invariants of etale covers are determined by their Taubes' type. Since the degree 1 map has positive sign, and two of (nontrivial) double covers have positive sign and one has negative sign, we have

$$\sum_{d>0} GW_1^{\mathrm{loc}}(F_2, d) \, t_{F_2}^d = \sum_{d>0} \frac{1}{d} \left[\sigma(d) - 2\,\sigma\!\left(\frac{d}{2}\right) \right] t_{F_2}^d$$

$$= \int \frac{G(t_{F_2}) - 2G(t_{F_2}^2)}{t_{F_2}} \, dt_{F_2}$$

(see Proposition 4.4 of [**12**]).

References

[1] E. Arbarello, M. Cornalba, P. Griffiths, & J. Harris, *Geometry of Algebraic Curves*, Vol. 1, Grundlehren der mathematischen Wissenschaften, **267**, Springer-Verlag, New York, 1985, MR 0770932, Zbl 0559.14017.

[2] M.F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. Ecole Norm. Sup., 1971, 47–62, MR 0286136, Zbl 0212.56402.

[3] I. Bauer & F. Catanese, *A volume maximizing canonical surface in 3-space*, preprint, math.AG/608020.

[4] W. Barth, K. Hulek, C. Peters, & A. Van de Ven, *Compact complex surfaces*, second ed., Springer-Verlag, Berlin-Heidelberg, 2004, MR 2030225, Zbl 1036.14016.

[5] M. Cornalba, *Moduli of curves and theta charateristics*, Lectures on Riemann Surfaces, 560–589, World Scientific, Singapore, 1989, MR 1082349, Zbl 0786.00021.

[6] R. Friedman & J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer-Verlag, Berlin-Heidelberg, 1994, MR 1288304, Zbl 0817.14017.

[7] A. Gathmann, *Gromov-Witten invariants of blow-ups*, J. Algebraic Geom. **10** (2001) 399–432, MR 1832328, Zbl 1080.14064.

[8] R. Gompf, *Nuclei of elliptic surfaces*, Topology **30** (1991) 479–511, MR 1113691, Zbl 0732.57010.

[9] P. Griffith & J. Harris, *Principles of algebraic geometry*, J. Wiley, New York, 1978, MR 1288523, Zbl 0408.14001.

[10] R. Hartshorne, *Algebraic Geometry*, Springer–Verlag, New York, 1977, MR 0463157, Zbl 0367.14001.

[11] J. Hu, *Gromov-Witten invariants of blow-ups along surfaces*, Compositio Math. **125** (2001) 345–352, MR 1818985, Zbl 1023.14029.

[12] E. Ionel & T.H. Parker, *The Gromov invariants of Ruan-Tian and Taubes*, Math. Res. Lett. **4** (1997) 521–532, MR 1470424, Zbl 0889.57030.

[13] _____, *Relative Gromov-Witten Invariants*, Ann. Math. **157** (2003) 45–96, MR 1954264, Zbl 1039.53101.

[14] _____, *The Symplectic Sum Formula for Gromov-Witten Invariants*, Ann. Math. **159** (2004) 935–1025, MR 2113018, Zbl 1075.53092.

[15] S. Ivashkovich & V. Shevchishin, *Pseudo-holomorphic curves and envelopes of meromorphy of two-spheres in $\mathbb{CP}^2$*, math.CV/9804014.

[16] M. Kontsevich & Yu. Manin, *Relations between the correlators of the topological sigma-model coupled to gravity*, Commun. Math. Phys, **196** (1998) 385–398, MR 1645019, Zbl 0946.14032.

[17] J. Lee, *Family Gromov-Witten Invariants for Kähler Surfaces*, Duke Math. J. **123** (2004) 209–233, MR 2060027, Zbl 1059.53070.

[18] _____, *Counting Curves in Elliptic Surfaces by Symplectic Methods*, Comm. Anal. Geom. **14** (2006) 107–134, MR 2230572.

[19] J. Lee & T.H. Parker, *Symplectic Gluing and Family Gromov-Witten Invariants*, Geometry and topology of manifolds, 147–172, Fields Inst. Commun., **47**, Amer. Math. Soc., Providence, RI, 2005, MR 2189931.

[20] J. Li & G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser. I, International Press, Cambridge, MA, 1998, MR 1635695, Zbl 0978.53136.

[21] Y. Ruan & G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995) 259–367, MR 1366548, Zbl 0860.58005.

[22] _____, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997) 455–516, MR 1483992, Zbl 0904.58066.

[23] C.H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension 4*, J. Differential Geom. **44** (1996) 818–893, MR 1438194, Zbl 0883.57020.

[24] M. Usher, *Standard surfaces and nodal curves in symplectic 4-manifolds*, preprint, math.SG/0407494.

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